Finding Time-Dependent Shortest Paths over Large Graphs

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ABSTRACT

The spatial and temporal databases have been studied widely and intensively over years. In this paper, we study how to answer queries of finding the best departure time that minimizes the total travel time from a place to another, over a road network, where the traffic conditions dynamically change from time to time. We study a generalized form of this problem, called the time-dependent shortest-path problem. A time-dependent graph G_T is a graph that has an edge-delay function, $w_{i,j}(t)$, associated with each edge (v_i, v_j), to be stored in a database. The edge-delay function $w_{i,j}(t)$ specifies how much time it takes to travel from node v_i to node v_j, if it departs from v_i at time t. A user-specified query is to ask the minimum-travel-time path, from a source node, v_s, to a destination node, v_e, over the time-dependent graph, G_T, with the best departure time to be selected from a time interval T. We denote this user query as $\mathsf{LTT}(v_s, v_e, T)$ over G_T. The challenge of this problem is the added complexity due to the time dependency in the time-dependent graph. That is, edge delays are not constants, and can vary from time to time. In this paper, we propose a novel algorithm to find the minimum-travel-time path with the best departure time for a $\mathsf{LTT}(v_s, v_e, T)$ query over a large graph G_T. Our approach outperforms existing algorithms in terms of both time complexity in theory and efficiency in practice. We will discuss the design of our algorithm, together with its correctness and complexity. We conducted extensive experimental studies over large graphs and will report our findings.

1. INTRODUCTION

Due to the increasing interest in the dynamic management of transportation systems, there are needs to find shortest paths over a large graph (e.g., a road network), where the weights (or delays) associated with edges dynamically change over time (*time-dependency*). Transportation systems, which can provide real-time traffic information (used to calculate edge delays) to users, include the Vehicle Information and Communication System[1] (VICS) and the European Traffic Message Channel[2] (TMC). The former operates in Japan and the latter operates in most European countries, North America, and Australia. Together with road networks available as large graphs[3], when such traffic information is available and the (periodical) traffic patterns are known over a long time period, it becomes possible to provide users with services, such as "how to travel from a place in a city to another place in another city as fast as possible", by taking "rush hour" into consideration.

Consider tourism as an application. Suppose a group of people wants to visit several places in several cities. When such road traffic information is available, the group wants to know whether they can travel to the next place *faster* (*spending less travel time on the way*), if they depart from a place later to avoid rush hour. In a similar fashion, consider a logistic company that delivers products for their customers using trucks. A truck may travel to a place with less travel time, if it stays somewhere for some time, say 3 hours. In this case, the company can utilize the 3 hours to deliver products to nearby customers where possible with this truck.

Assume a road network is stored as a large graph with the traffic information in a database. Such a query can be specified as follows. Given a source v_s and a destination v_e, over the graph, and a time window T for consideration of departure from v_s, find the best time within T to depart from v_s, and identify the path along which one can arrive at v_e with the minimum travel time.

In this paper, we study the generalized form of this query, called *time-dependent shortest-path (TDSP) problem*: to find the optimal path (with the minimum travel time) from a source to a destination, over a *time-dependent graph*, when the starting time (departure time from the source) is selected from a user-given starting-time interval. The time-dependent graph is a graph that has an edge-delay (travel time from v_i to v_j) function $w_{i,j}(t)$, w.r.t. departure time t from v_i, for each edge (v_i, v_j). TDSP problem was studied to either find approximate answers with discrete-time approaches [1, 2] or find optimal answers with continuous-time approaches [20, 15].

We focus on finding optimal answers for the TDSP problem using a continuous-time approach with less time/space complexity. We consider a specific class of graphs, called *FIFO* time-dependent graphs (refer to Section 5.1), as well as general time-dependent graphs. Our approach can handle arbitrary edge-delay functions, and allows waiting on nodes in order to minimize the travel time.

Contributions of this paper: (1) We propose a novel algorithm to find optimal answers for the TDSP problem. Our algorithm can handle both undirected and directed time-dependent graphs, and both *FIFO* and non-*FIFO* time-dependent graphs. (2) We show that the time complexity of our algorithm is $O((n \log n + m)\alpha(T))$ and

[1] `http://www.vics.or.jp/english/index.html/`

[2] `http://www.tmcforum.com/`

[3] `http://maps.google.com/`

EDBT'08, March 25–30, 2008, Nantes, France.

the space complexity is $O((n+m)\alpha(T))$, where n is the number of nodes, m is the number of edges, and $\alpha(T)$ is the cost required for each function (defined in interval T) operation. Our algorithm can be used to handle large time-dependent graphs. (3) We discuss storage model and implementation, and show that our approach can be easily implemented in a database system. (4) We conducted extensive performance studies, and we show that our algorithm outperforms existing solutions in terms of efficiency and effectiveness.

Organization: Section 2 gives the problem statement. Section 3 introduces existing solutions to the time-dependent shortest-path problem. We give an overview of our algorithm in Section 4, and give the details in Section 5, including a running example, discussions on the correctness and time/space complexity of our algorithm, implementation details, and how to handle non-*FIFO* graphs. We give the experimental results in Section 6. Section 7 discusses the related work. Finally, we conclude our paper in Section 8.

2. PROBLEM DEFINITION

Definition 2.1: (**Time-Dependent Graph**) A *time-dependent graph* is defined as $G_T(V, E, W)$ (or G_T for short): $V = \{v_i\}$ is a set of nodes; $E \subseteq V \times V$ is a set of edges; W is a set of positive-valued functions. For every edge $(v_i, v_j) \in E$, there is a function $w_{i,j}(t) \in W$, where t is a time variable in a time domain $\mathcal{T}$. An *edge-delay function* $w_{i,j}(t)$ specifies how much time it takes to travel from v_i to v_j, if departing v_i at time t. □

In this paper, we concentrate on finding the *least total travel time* (LTT) from source node v_s to destination node v_e when the *starting time* t (departure time from the source), can be selected in a user-given *starting-time interval* $T = [t_s, t_e] \subseteq \mathcal{T}$. Such a query is called an LTT*query*, denoted as $\mathsf{LTT}(v_s, v_e, T)$.

Note the *travel time* is the *arrival time* minus the *starting time*. In order to find LTT, we allow *waiting time*, denoted as $\varpi(v_i)$, at each node v_i. That is, when arriving at node v_i, we can wait for a time period $\varpi(v_i)$ if LTT can be minimized. Below, let $\mathsf{arrive}(v_i)$ and $\mathsf{depart}(v_i)$ denote the *arrival time* at node v_i and the *departure time* from node v_i, respectively. For each node v_i, we have

$$\mathsf{depart}(v_i) \quad = \quad \mathsf{arrive}(v_i) + \varpi(v_i). \tag{1}$$

Let $\mathsf{p} = (v_1, v_2)(v_2, v_3)\cdots(v_{k-1}, v_k)$ be a fixed path with waiting time $\varpi(v_i)$ at node v_i. For a fixed starting time t,

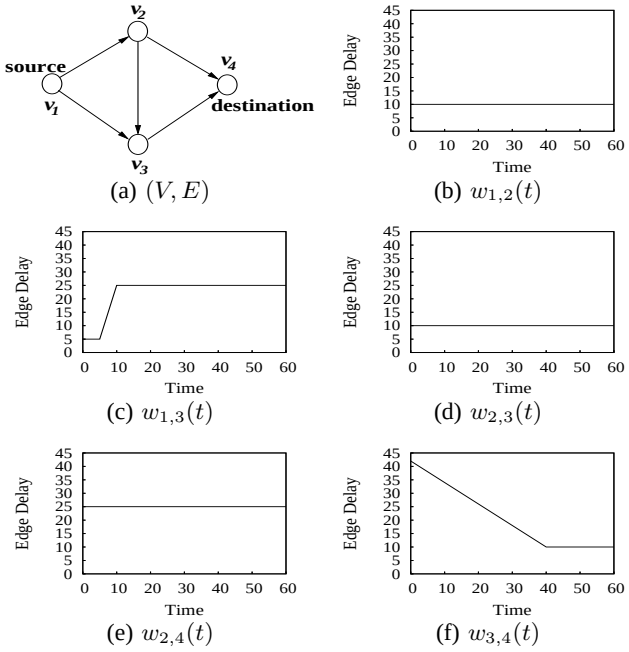
$$\begin{aligned}
\mathsf{arrive}(v_1) &= t &(2)\\
\mathsf{arrive}(v_2) &= \mathsf{depart}(v_1) + w_{1,2}(\mathsf{depart}(v_1))\\
&\cdots\\
\mathsf{arrive}(v_k) &= \mathsf{depart}(v_{k-1}) + w_{k-1,k}(\mathsf{depart}(v_{k-1})) &(3)\\
g_{\mathsf{p}}(t) &= \mathsf{arrive}(v_k). &(4)
\end{aligned}$$

$g_{\mathsf{p}}(t)$ above is the *arrival-time function*, representing the arrival time from v_1 to v_k along path p, possibly waiting at some nodes on this path, if departing from v_1 at starting time t. The *travel-time function* along path p is thus $g_{\mathsf{p}}(t) - t$. We formally define the time-dependent shortest-path (TDSP) problem as follows.

Definition 2.2: (**TDSP Problem**) Given a time-dependent graph $G_T(V, E, W)$ and an LTT *Query* $\mathsf{LTT}(v_s, v_e, T)$, where $v_s, v_e \in V$, and $T \subseteq \mathcal{T}$ is a starting-time interval, the *Time-Dependent Shortest-Path (TDSP) problem* is to minimize LTT:

$$g_{\mathsf{p}^*}(t^*) - t^* = \min_{\mathsf{p}, \varpi(\cdot), t} \{g_{\mathsf{p}}(t) - t\} \tag{5}$$

finding a v_s-v_e path p^* with waiting time $\varpi^*(v_i)$ at v_i, along which the best starting time t^* results in the minimum travel time $g_{\mathsf{p}}(t) - t$ among all starting times $t \in T$ and over all v_s-v_e paths p's. □



(a) (V, E) (b) $w_{1,2}(t)$ (c) $w_{1,3}(t)$ (d) $w_{2,3}(t)$ (e) $w_{2,4}(t)$ (f) $w_{3,4}(t)$

Figure 1: A Time-Dependent Graph $G_T(V, E, W)$

Example 2.1: A road network can be modelled as a time-dependent graph $G_T(V, E, W)$ in Fig. 1. Fig. 1 (a) shows its graph structure (V, E), with four nodes and five edges. The edge-delay functions for the edges, (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), and (v_3, v_4), are shown in Fig. 1 (b), (c), (d), (e), and (f), respectively.

For query $\mathsf{LTT}(v_1, v_4, [0, 60])$, $\mathsf{p}^* = (v_1, v_2)(v_2, v_3)(v_3, v_4)$ is the optimal v_1-v_4 path. Along p^* with no waiting time required at any node, the best starting time $t^* = 20$ results in the minimum travel time $g_{\mathsf{p}^*}(t^*) - t^* = 30$ (which will be further explained in Section 5.3 as a running example of our solution). □

In the following part, we focus on the TDSP problem, i.e., for query $\mathsf{LTT}(v_s, v_e, T)$ over a time-dependent graph $G_T(V, E, W)$, finding the optimal v_s-v_e path p^*, waiting times $\varpi^*(\cdot)$, and starting time t^*, to minimize LTT. Note T is a continuous time interval, and t^* can be any time point within this interval.

3. EXISTING SOLUTIONS

In this section, we discuss three types of algorithms for the TDSP problem, to answer query $\mathsf{LTT}(v_s, v_e, T)$ over a time-dependent graph $G_T(V, E, W)$. They are discrete-time algorithms [17, 2, 1], BELLMAN-FORD based algorithm [20], and extended A* algorithm [15]. The discrete-time algorithms find an approximate LTT solution, and both BELLMAN-FORD and A* algorithms find the optimal (minimized) LTT. The main challenge to find the optimal LTT over $G_T(V, E, W)$ is, because edge delays are different functions of departure times, the v_s-v_e path with the least total travel time changes in a complicated manner as the starting time changes.

Discrete-Time Algorithms: The discrete-time approaches have been well-studied. To the best of our knowledge, the most efficient one, denoted as DOT, was presented in [2]. They find approximate LTT by globally discretizing time interval into time points. In brief, given a graph $G_T(V, E, W)$, a discrete-time approach discretizes the starting-time interval $T = [t_s, t_e]$ into k time points evenly, and constructs a static graph $G'_T(V', E', W')$ by making k copies of each node and each edge, respectively. Thus, $|V'| =$

$k|V|$, $|E'| = k|E|$, and edge delay W' is static. For each edge $(v'_i, v'_j) \in E'$, edge delay $w'_{i,j}$ is equal to the value of $w_{i,j}(t)$ on a time point. The TDSP problem on $G_T(V, E, W)$ can be solved as a static single-source shortest path problem on $G'_T(V', E', W')$, whose size is enlarged k times. Its solution can be used to approximate LTT over $G_T(V, E, W)$.

There are two fundamental drawbacks inherent in discrete-time approaches. First, the difference between the LTT obtained using a discrete-time approach and the optimal LTT, called LTT *error*, is very sensitive to parameter k, and is unbounded. This is because the optimal starting time t^* for $\mathsf{LTT}(v_s, v_e, T)$ can be always between any two of the k time points, and the LTT error is generated in an accumulative way along v_s-v_e paths. Second, increasing k deteriorates the efficiency of discrete-time approaches, since G'_T is k times larger than G_T.

Bellman-Ford Based Algorithm: Orda and Rom [20] proposed a continuous-time algorithm to solve the TDSP problem. We call it OR algorithm by taking the initials of Orda and Rom. Algorithm OR takes time-dependent graph $G_T(V, E, W)$ and query $\mathsf{LTT}(v_s, v_e, T)$ as the input. It is outlined below.

1: **for all** $v_l \in V$ **do** $g_l(t) \leftarrow \infty$ for $t \in T$;
2: **for all** $(v_k, v_l) \in E$ **do** $h_{k,l}(t) \leftarrow \infty$ for $t \in T$;
3: $g_s(t) \leftarrow t$ for $t \in T$;
4: **repeat**
5: **for all** $(v_k, v_l) \in E$ **do** $h_{k,l}(t) \leftarrow g_k(t) + w_{k,l}(g_k(t))$;
6: **for all** $v_l \in V$ **do** $g_l(t) \leftarrow \min_{v_k \in N(v_l)}\{h_{k,l}(t)\}$;
7: **until** all functions $g_l(t)$ are unchanged
8: **return** $(t^* \leftarrow \mathrm{argmin}_{t \in T}\{g_e(t) - t\}, \mathsf{p}^*)$;

OR generalizes the BELLMAN-FORD shortest-path algorithm. Let function $g_l(t)$ be the *earliest arrival time* at node v_l, from source v_s, for starting time t, and let function $h_{k,l}(t)$ be the *earliest arrival time* at v_l, from source v_s via edge (v_k, v_l), for starting time t. It first initializes $g_l(t)$ and $h_{k,l}(t)$ functions (line 1-3), and then repeatedly updates $g_l(t)$ and $h_{k,l}(t)$ until they converge to the correct values (line 4-7). Finally (line 8), it returns the best starting time t^*, and the optimal v_s-v_e path p^*, as the answer to $\mathsf{LTT}(v_s, v_e, T)$. p^* is constructed based on $g_l(t)$ and $h_{k,l}(t)$ functions (refer to [20]).

The time complexity of Algorithm OR is $O(nm\alpha(T))$, where $\alpha(T)$ is the time required in a function operation in interval T, $n = |V|$, and $m = |E|$. The high time complexity makes it infeasible for OR to work on large or dense time-dependent graphs. We outline the reasons for its high time complexity below.

OR takes a strategy of determining paths toward destination v_e while refining the arrival-time functions, $g_i(t)$, in the whole interval T. We call such an algorithm a *path-selection and time-refinement* approach. The *path-selection* is accomplished implicitly in line 5, attempting to arrive at v_l earlier via edge (v_k, v_l). The *time-refinement* is done in line 6, updating arrival-time function $g_l(t)$ using $h_{k,l}(t)$. The interweavement of path-selection and time-refinement *in the whole interval* T makes functions, $g_l(t)$ and $h_{k,l}(t)$, converge slowly, possibly in n iterations of line 4-7. Actually, after some iterations, $g_l(t)$ might have converged in a subinterval of T, but Algorithm OR cannot recognize this, and still needs to recalculate $g_l(t)$ and $h_{k,l}(t)$ in the whole interval T.

A* Algorithm: Kanoulas et al. in [15] gave an extension to A* algorithm for the TDSP problem. We denote it as KDXZ by taking the initials from the authors in [15]. The main idea is to maintain a priority queue $\mathcal{Q}$ of all paths to be expanded. Let p_k be a path from source v_s to a node v_k. Note: there are possibly multiple paths from v_s to v_k in G_T, and all of them may be maintained in $\mathcal{Q}$ at the same time. Each distinct v_s-v_k path p_k is associated with a function, $f_{\mathsf{p}_k}(t) = g_{\mathsf{p}_k}(t) + d_{k,e} - t$. Here, $g_{\mathsf{p}_k}(t)$ is the arrival time from source v_s to v_k along path p_k for starting time t; $d_{k,e}$ is a lower bound estimation of the travel time from v_k to destination v_e; $f_{\mathsf{p}_k}(t)$ is the estimated travel time from source v_s to destination v_e along path p_k for starting time t. In each iteration, it picks the path p_i from the priority queue $\mathcal{Q}$ to expand, such that $\min_t\{f_{\mathsf{p}_i}(t)\}$ is the minimum among all paths p_k's in $\mathcal{Q}$. Each path p_j, extended from p_i with one more edge (v_i, v_j), will be added into the priority queue $\mathcal{Q}$ for further expansion, and path p_i will be deleted from $\mathcal{Q}$. This process will terminate when the first v_s-v_e path p_e is picked from $\mathcal{Q}$. Note KDXZ assumes no waiting is allowed.

KDXZ is also a *path-selection and time-refinement* approach. The *path-selection* is done explicitly in the path extension from p_k to p_l, followed by the *time-refinement* done in the computation of $g_{\mathsf{p}_l}(t)$ and $f_{\mathsf{p}_l}(t)$. The *path-selection* and the *time-refinement* here are coupled even more closely than those in OR. Resultingly, in the worst case, all v_s-v_e paths are enumerated, and the time/space complexity of KDXZ is exponential w.r.t. the size of G_T.

Algorithm KDXZ is efficient only when estimation can assist pruning the search space effectively, and v_s and v_e are closed to each other in graph G_T. It is difficult to find such estimation $d_{k,e}$ in general graphs, and it is infeasible to use KDXZ to handle large time-dependent graphs, where v_e may be far away from v_s.

Remark 3.1: ***(About Functions)*** *While discrete-time algorithms avoid the representation and operations of functions,* BELLMAN-FORD *based algorithm [20],* A* *algorithm [15], and ours find the optimal* LTT *based on four basic function operations:* FUNCTION INVERSE, $f^{-1}(a) \triangleq \max\{t | f(t) = a\}$, LINEAR COMBINATION, $a \cdot f(t) + b \cdot g(t)$, FUNCTION COMPOUND, $f(g(t))$, *and* MINIMUM *of two functions,* $\min\{f(t), g(t)\}$. *[20] considers a general class of functions from a theoretical view, whereas [15] focuses on piecewise linear functions with the consideration of the cost to manipulate such functions. In this paper, we will show our approach can also handle a general class of functions as Algorithm* OR *in [20] does. Sharing the same concerns with [15], we will focus on piecewise-linear functions regarding implementations and performance studies in this paper.* □

4. NEW DIJKSTRA BASED ALGORITHM

In the following part, we first focus on answering $\mathsf{LTT}(v_s, v_e, T)$ queries in an *FIFO* (*First-In and First-Out*, Definition 5.1) time-dependent graph G_T, where no waiting time is needed in optimal solutions (Theorem 5.1). We will discuss how to deal with general graphs in Section 5.7. We assume G_T is a *directed graph*. With minor changes, our algorithm can handle *undirected graphs* as well.

We propose a new algorithm by decoupling *path-selection* and *time-refinement* in the starting-time interval T. We show that answering a query $\mathsf{LTT}(v_s, v_e, T)$ over a graph G_T can be done in two steps. In the first step, we focus on time-refinement, i.e., for every node $v_i \in V$, to compute the *earliest arrival time*, $g_i(t)$, departing from v_s at any starting time $t \in T$. We call $g_i(t)$ the v_s-v_i *earliest arrival-time function* in the following part. Based on the earliest arrival-time functions computed, the best starting time t^* with the minimum v_s-v_e travel time, $g_e(t^*) - t^* = \min_t\{g_e(t) - t\}$, can be identified. In the second step, we select one of the paths from v_s to v_e, which matches the optimal travel time $g_e(t^*) - t^*$.

As an example to illustrate the main ideas, consider the query $\mathsf{LTT}(v_1, v_4, T)$ over G_T (Fig. 1) in Example 2.1, where $T = [0, 60]$. In the first step, we compute the earliest arrival-time functions, $g_1(t)$, $g_2(t)$, $g_3(t)$, and $g_4(t)$, for the four nodes, v_1, v_2, v_3, and v_4. The earliest arrival-time function, $g_4(t)$, and its corresponding travel time function, $g_4(t) - t$, from source v_1 to destina-

Notation	Meaning
$G_T(V, E, W)$	time-dependent graph (or G_T for short)
n, m	number of nodes $\|V\|$, number of edges $\|E\|$
$w_{i,j}(t)$	edge-delay function for $(v_i, v_j) \in E$
v_s, v_e, T	source, destination, starting-time interval
p^*	optimal path from v_s to v_e
t^*	optimal starting time
$\varpi^*(v_i)$	optimal waiting time at node v_i
$g_i(t)$	v_s-v_i earliest arrival-time function
$g_{\mathsf{p}}(t)$	arrival-time function (along path p)
$\alpha(T)$ or $\alpha(\|T\|)$	time/space required to maintain a function or to manipulate a function operation over time interval T

Table 1: Important Notations

tion v_4, are shown in Fig. 2 (a) and (b), respectively. As shown in Fig. 2 (b), the least total travel time is 30, and the best starting time is $t^* = 20$, which is a starting time to arrive at v_4 within the minimum travel time 30. In the second step, we identify the optimal path as $\mathsf{p}^* = (v_1, v_2)(v_2, v_3)(v_3, v_4)$. Note in Fig. 2, $g_4(t)$ and $g_4(t) - t$ are given in a subinterval $[0, 30]$ of $T = [0, 60]$, because if starting from v_1 later than 30, it will arrive at v_4 later than 60, and thus some edge-delay functions are undefined.

The first step is the dominating factor in terms of computational cost. It needs to compute the earliest arrival-time function $g_i(t)$ for every node $v_i \in V$, as given in Equation (6).

$$g_i(t) = \min_{v_j \in N(v_i), \varpi(v_j)} \{(g_j(t) + \varpi(v_j)) + w_{i,j}(g_j(t) + \varpi(v_j))\} \quad (6)$$

Here, $N(v_i)$ is a set of neighbors of v_i that can reach v_i in graph G_T, i.e., $N(v_i) = \{v_j | (v_j, v_i) \in E\}$.

The challenge of computing Equation (6) is due to the edge-delay functions. The edge delays are not constants, and can vary for different starting times. Therefore, the optimal v_s-v_e path may be different for different starting time. In a continuous starting-time interval, there are infinite different starting-time values. It is challenging to select the best starting time t^* and the optimal v_s-v_e path from an infinite number of possible starting times and an exponential number of v_s-v_e paths, respectively.

Below, we show our solution TWO-STEP-LTT decouples the two things, namely, *path-selection* and *time-refinement*. We design a DIJKSTRA-based algorithm for the first step (time-refinement), and a linear-time algorithm for the second step (path-selection).

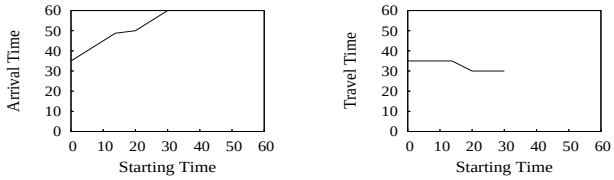
Outline of Two-Step-LTT (Algorithm 1): The main part of our two-step algorithm is given in Algorithm 1. We call it TWO-STEP-LTT. As shown in Algorithm 1, it takes four input parameters: time-dependent graph G_T, source v_s, destination v_e, and starting-time interval T (a query $\mathsf{LTT}(v_s, v_e, T)$ over G_T). The first step, *timeRefinement* (Algorithm 3), computes the earliest arrival-time functions $g_i(t)$, for nodes v_i in G_T, in line 1. The condition in line 2 checks whether there is a path from v_s to v_e. The optimal starting time t^* is identified in line 3. For the second step, it calls *pathSelection* (Algorithm 2) to find a path p^* which matches the arrival time $g_e(t^*)$ for the best starting time t^* in line 4. Finally, it returns path p^* together with the best starting time t^*. We outline the main ideas behind the two steps below.

Dijkstra-Based Time-Refinement (Algorithm 3): In the first step, we compute the earliest arrival-time function $g_i(t)$, for every node v_i in V. Like the DIJKSTRA algorithm (for the static shortest-path problem) which expands a set of nodes, we refine arrival-time functions, $g_i(t)$, incrementally in the given starting-time interval

Algorithm 1 TWO-STEP-LTT $(G_T(V, E, W), v_s, v_e, T)$

Input: a time-dependent graph G_T, a query $\mathsf{LTT}(v_s, v_e, T)$ - source v_s, destination v_e, and starting-time interval $T = [t_s, t_e]$;
Output: optimal v_s-v_e path p^*, and optimal starting time t^*.

1: $\{g_i(t)\} \leftarrow$ *timeRefinement* (G_T, v_s, v_e, T);
2: **if** $\neg(g_e(t) = \infty$ for the entire $[t_s, t_e])$ **then**
3: $\quad t^* \leftarrow \text{argmin}_{t \in T}\{g_e(t) - t\}$;
4: $\quad \mathsf{p}^* \leftarrow$ *pathSelection* $(G_T, \{g_i(t)\}, v_s, v_e, t^*)$;
5: $\quad$ **return** (t^*, p^*);
6: **else return** $\emptyset$;



(a) $g_4(t)$ (b) $g_4(t) - t$

Figure 2: An earliest arrival-time function and its corresponding travel-time function

$T = [t_s, t_e]$. For every node $v_i \in V$, let $I_i = [t_s, \tau_i] \subseteq T$ be a starting-time subinterval, where time point $\tau_i \in T = [t_s, t_e]$. By "incrementally", we mean: we refine the earliest arrival-time function $g_i(t)$ by extending I_i to a larger starting-time subinterval $I_i' = [t_s, \tau_i'] \subseteq T$, for $\tau_i' > \tau_i$, and computing $g_i(t)$ in $[\tau_i, \tau_i']$. We say *function $g_i(t)$ is well-refined* in a starting-time subinterval I_i, if it specifies the earliest arrival time at v_i from v_s for any starting time $t \in I_i$. It is possible that $I_i \neq I_j$ for $v_i \neq v_j$.

In our algorithm, we promise function $g_i(t)$ is well-refined in I_i for each $v_i \in V$. In every iteration, we select a node v_i, and expand its starting-time subinterval from I_i to I_i', in which function $g_i(t)$ is well-refined. We update the arrival-time function $g_j(t)$ for every neighbor, v_j, of node v_i in the starting-time subinterval $I_i' - I_i = [\tau_i, \tau_i']$. Then we let $I_i = I_i'$, and repeat this time-refinement process, namely DIJKSTRA-based time-refinement, till function $g_e(t)$ is well-refined, for destination v_e, in the entire starting-time interval T. The best starting time, $t^* \in T$, is identified as $\text{argmin}_{t \in T}\{g_e(t) - t\}$, which minimizes $g_e(t) - t$.

Fast Path-Selection (Algorithm 2): The optimal v_s-v_e path p^* is computed using the *pathSelection* algorithm (Algorithm 2), which takes five inputs: graph G_T, all the earliest arrival-time functions $\{g_i(t)\}$, the optimal starting time $t^* \in T$, source v_s, and destination v_e. To select the path p^* from v_s to v_e, we determine the predecessor of every node on p^* backward from v_e to v_s based on $\{g_i(t)\}$ and t^*. The predecessor of v_j is determined as v_i, if $g_j(t^*) = g_i(t^*) + w_{i,j}(g_i(t^*))$, for $(v_i, v_j) \in E$. It means that the arrival time at v_j, $g_j(t^*)$, is the arrival time at v_i, $g_i(t^*)$, plus the edge delay from v_i to v_j (we assume there is no waiting time here).

In *pathSelection* (Algorithm 2), initially, we set v_j as destination v_e, and the optimal path p^* empty (line 1-2). In the while loop, we iteratively find a predecessor v_i of v_j and add (v_i, v_j) into p^* till p^* reaches the source v_s (line 3-7). The correctness of *pathSelection* is straightforward. Its time complexity is $O(m\alpha(T))$, where $m = |E|$ and $\alpha(T)$ is the time required for each function operation, because each edge can be examined in line 5 at most once.

In the following, we will focus on the first step of TWO-STEP-LTT, namely, time-refinement.

Remark 4.1: *Comparing Algorithm* TWO-STEP-LTT *with Algo-*

Algorithm 2 *pathSelection* ($G_T(V, E, W)$, $\{g_i(t)\}$, v_s, v_e, t^*)

Input: a time-dependent graph G_T, the set of earliest arrival-time functions $g_i(t)$ for all nodes $v_i \in V$, source node v_s, destination node v_e, and the optimal starting time t^*;
Output: an optimal v_s-v_e path p^* for starting time t^*.

1: $v_j \leftarrow v_e$;
2: $\mathsf{p}^* \leftarrow \varnothing$;
3: **while** $v_j \neq v_s$ **do**
4: **for each** $(v_i, v_j) \in E$ **do**
5: **if** $g_i(t^*) + w_{i,j}(g_i(t^*)) = g_j(t^*)$ **then**
6: $v_j \leftarrow v_i$; **break**;
7: $\mathsf{p}^* \leftarrow (v_i, v_j) \cdot \mathsf{p}^*$;
8: **return** p^*;

rithm OR, $h_{k,l}(t)$ *functions are absent in our* TWO-STEP-LTT, *and* $g_i(t)$ *functions share the same meanings in both. The absence of* $h_{k,l}(t)$ *functions in* TWO-STEP-LTT *does not add more complexity to the construction of* p^* *(Algorithm 2). As shown in Section 5 (Algorithm 3), we can use* $g_i(t)$ *functions solely to answer query* $\mathsf{LTT}(v_s, v_e, T)$ *with lower time/space complexity.* □

5. TIME-REFINEMENT

In this section, given $G_T(V, E, W)$ and query $\mathsf{LTT}(v_s, v_e, T)$, we focus on the first step of TWO-STEP-LTT, i.e., time-refinement. By time-refinement, we mean to compute and refine the earliest arrival-time function $g_i(t)$ for every node v_i in V.

First, we introduce a special class of time-dependent graphs, called *FIFO* (*First In, First Out*) graphs [20]. Second, we discuss our DIJKSTRA-based algorithm *timeRefinement* (Algorithm 3) to compute the earliest arrival-time function $g_i(t)$ for every node v_i, for answering a query $\mathsf{LTT}(v_s, v_e, T)$, in a *FIFO* graph $G_T(V, E, W)$. It is based on the incremental time-refinement of functions $g_i(t)$ for nodes v_i in starting-time interval T. Third, we explain our algorithm using an example. Fourth, we prove the correctness of our algorithm, and give its time/space complexity. Finally, we discuss some implementation details, and show how our DIJKSTRA-based algorithm can also work on general non-*FIFO* graphs.

5.1 FIFO Graphs

FIFO property of an edge (v_i, v_j), in G_T, suggests that if departing earlier from v_i, one arrives earlier at v_j.

Definition 5.1: (**FIFO**) Time-dependent graph $G_T(V, E, W)$ is a *FIFO graph*, iff every edge (v_i, v_j) has *FIFO property*. An edge (v_i, v_j) has *FIFO* property, iff $w_{i,j}(t_0) \leq t_\Delta + w_{i,j}(t_0 + t_\Delta)$ for $t_\Delta \geq 0$, or $t_1 + w_{i,j}(t_1) \leq t_2 + w_{i,j}(t_2)$ for $t_1 \leq t_2$. □

Theorem 5.1: (**No Waiting in FIFO Graphs**) *For a given query* $\mathsf{LTT}(v_s, v_e, T)$ *on a FIFO time-dependent graph* G_T*, there exists an optimal* v_s-v_e *path* p^* *along which the optimal waiting time* $\varpi^*(v_i) = 0$ *for every* v_i *on* p^*. □

Proof Sketch: Let v_i be a node on optimal path p^*, s.t. $\varpi^*(v_i) > 0$, and v_j be v_i's successor on p^*. Let $t_i = \mathsf{arrive}(v_i)$ and $t_j = \mathsf{arrive}(v_j)$ be the arrival time at v_i and v_j, respectively, along p^* for starting time t^*. From *FIFO* property, we have $t_i + w_{i,j}(t_i) \leq (t_i + \varpi^*(v_i)) + w_{i,j}(t_i + \varpi^*(v_i)) = t_j$. That is, the arrival time at v_j without waiting on v_i (i.e., $t_i + w_{i,j}(t_i)$) is no later than the arrival time at v_j with waiting time $\varpi^*(v_i)$ on v_i (i.e., t_j). By induction, we can prove if $\varpi^*(v_i) = 0$ for each node v_i, the travel time at v_e along p^* do not increase. Details are omitted. □

Algorithm 3 *timeRefinement* ($G_T(V, E, W)$, v_s, v_e, T)

Input: a time-dependent graph G_T, a query $\mathsf{LTT}(v_s, v_e, T)$ - source v_s, destination v_e, and starting-time interval $T = [t_s, t_e]$;
Output: $\{g_i(t) | v_i \in V\}$ - all earliest arrival-time functions.

1: $g_s(t) \leftarrow t$ for $t \in T$; $\tau_s \leftarrow t_s$;
2: **for each** $v_i \neq v_s$ **do**
3: $g_i(t) \leftarrow \infty$ for $t \in T$; $\tau_i \leftarrow t_s$;
4: Let Q be a priority queue initially containing pairs, $(\tau_i, g_i(t))$, for all nodes $v_i \in V$, ordered by $g_i(\tau_i)$ in ascending order;
5: **while** $|Q| \geq 2$ **do**
6: $(\tau_i, g_i(t)) \leftarrow dequeue(Q)$;
7: $(\tau_k, g_k(t)) \leftarrow head(Q)$;
8: $\Delta \leftarrow \min\{w_{f,i}(g_k(\tau_k)) \mid (v_f, v_i) \in E\}$;
9: $\tau_i' \leftarrow \max\{t \mid g_i(t) \leq g_k(\tau_k) + \Delta\}$;
10: **for each** $(v_i, v_j) \in E$ **do**
11: $g_j'(t) \leftarrow g_i(t) + w_{i,j}(g_i(t))$ for $t \in [\tau_i, \tau_i']$;
12: $g_j(t) \leftarrow \min\{g_j(t), g_j'(t)\}$ for $t \in [\tau_i, \tau_i']$;
13: $update(Q, (\tau_j, g_j(t)))$;
14: $\tau_i \leftarrow \tau_i'$;
15: **if** $\tau_i \geq t_e$ **then**
16: **if** $v_i = v_e$ **then**
17: **return** $\{g_i(t) | v_i \in V\}$;
18: **else**
19: $enqueue(Q, (\tau_i, g_i(t)))$;
20: **return** $\{g_i(t) | v_i \in V\}$.

Theorem 5.1 implies that, to find an optimal solution to query $\mathsf{LTT}(v_s, v_e, T)$ over a *FIFO* graph G_T, we can safely assume waiting time $\varpi(v_i) = 0$ for each node $v_i \in V$, although waiting at nodes is allowed. The *road network model* studied in [15] is a FIFO graph (we will explain this in details in the appendix). Thus, waiting is not needed in road networks.

5.2 Time-Refinement for FIFO Graphs

In this subsection, we discuss how to process DIJKSTRA-based time-refinement for *FIFO* graphs, i.e., how to refine the earliest arrival-time function $g_i(t)$ in the starting-time interval T for every node v_i in G_T. The *timeRefinement* algorithm is outlined in Algorithm 3. It takes four parameters as the input: time-dependent graph $G_T(V, E, W)$, source node v_s, destination node v_e, and starting-time interval $T = [t_s, t_e]$. Here, by time-refinement we mean two things: *arrival-time function refinement* of $g_i(t)$ and *starting-time interval refinement* of the starting-time subinterval $I_i = [t_s, \tau_i]$, for every node v_i in G_T. Recall $I_i = [t_s, \tau_i]$ denotes the starting-time subinterval, on which function $g_i(t)$ is well-refined (or $g_i(t)$ specifies the earliest v_s-v_i arrival time for any starting time $t \in I_i$).

Initially, for source v_s, $g_s(t)$ and τ_s are initialized: $g_s(t) \leftarrow t$ and $\tau_s \leftarrow t_s$ (line 1). It means a trivial case: if it departs from source v_s at any time t_0, it will arrive at the same node v_s at the same time t_0, and its travel time is $g_s(t_0) - t_0 = 0$. For all other nodes, $v_i \neq v_s$, the earliest arrival-time functions, $g_i(t)$, are initialized as $g_i(t) \leftarrow \infty$, which means that they are undetermined yet, and all τ_i are initialized as $\tau_i \leftarrow t_s$ (line 2-3). For each node $v_i \in V$, $g_i(t)$ is ensured to be the earliest arrival time (well-refined) in $I_i = [t_s, \tau_i]$, which is a *loop invariant* in *timeRefinement* algorithm. Note: initially I_i is an empty subinterval.

Our algorithm uses a priority queue, Q, which initially contains pairs $(\tau_i, g_i(t))$'s for all nodes $v_i \in G_T$ in the ascending order of $g_i(\tau_i)$. The top pair in Q is $(\tau_s, g_s(t))$ initially. The while statement (line 5-19) conducts time-refinement for every node v_i

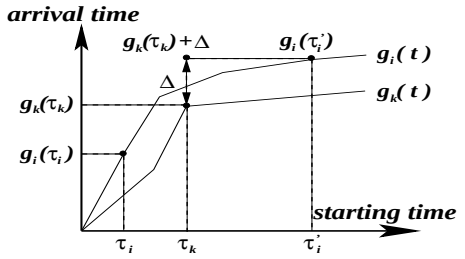


Figure 3: Starting-Time Interval Refinement

in G_T. It is ensured that the earliest arrival-time function $g_i(t)$ is well-refined in the starting-time subinterval $I_i = [t_s, \tau_i]$ for node v_i. The algorithm will terminate if $g_e(t)$, for destination v_e, is well-refined in the entire interval T (line 17), or Q contains no more than one pair (line 5). In every iteration in the while loop, the starting-time interval refinement is conducted in line 6-9 and line 14, and the arrival-time function refinement is conducted in line 10-13.

Next, we discuss starting-time interval refinement and arrival-time function refinement in details.

Starting-Time Interval Refinement: In every iteration, it first dequeues the top pair from Q, denoted as $(\tau_i, g_i(t))$ (line 6). After dequeuing, it will use the current top pair in Q, denoted as $(\tau_k, g_k(t))$, as the basis for starting-time interval refinement (line 7 - The operation $head(Q)$ retrieves the top pair but does not dequeue it from Q). Therefore, $g_i(\tau_i)$ is the earliest arrival time from source node v_s, followed by $g_k(\tau_k)$, among all pairs $(\tau_l, g_l(t))$'s in Q.

It is important to note that for any node v_f (except v_i), if the starting time is taken in $[\tau_f, t_e]$, it is impossible to arrive at v_f before the arrival time $g_k(\tau_k)$, from source node. The two reasons are given below. Let $(\tau_f, g_f(t))$ be in Q for node $v_f \in V$, and it arrives at node v_f at the arrival time $g_f(\tau_f)$ for starting time τ_f. First, $g_l(\tau_f) \geq g_k(\tau_k)$, because the sorting order in Q. Second, graph G_T is *FIFO*, and thus it arrives at v_f no earlier than time $g_k(\tau_k)$, if the starting time is taken in $[\tau_f, t_e]$. It can be formally proved based on the choices of $(\tau_i, g_i(t))$ and $(\tau_k, g_k(t))$ in Q, and the *FIFO* property of G_T (refer to Section 5.4).

Fix node v_i, and consider an edge $(v_f, v_i) \in E$ at time $g_k(\tau_k)$. If starting time t is taken in $[\tau_f, t_e]$, from the above discussion, it arrives at v_f no earlier than time $g_k(\tau_k)$. Suppose that it arrives at v_f at time $g_k(\tau_k)$. The minimum travel time from v_f to v_i can be computed as $\Delta \leftarrow \min\{w_{f,i}(g_k(\tau_k))|(v_f, v_i) \in E\}$ (line 8). Therefore, due to the *FIFO* property of G_T, next earliest possible arrival time from v_s to v_i via any edge (v_f, v_i) is $g_k(\tau_k) + \Delta$, if starting time $t \geq \tau_f$. We attempt to find the latest starting time t that satisfies $g_i(t) \leq g_k(\tau_k) + \Delta$, and set it as τ_i' (line 9). With the choice of τ_i', we can prove that (refer to Section 5.4) function $g_i(t)$ is well-refined in $[t_s, \tau_i']$, i.e., $g_i(t)$ is the earliest arrival time from v_s to v_i for starting time $t \in [t_s, \tau_i']$, because (the intuition) $g_i(t) \leq g_k(\tau_k) + \Delta$ for $t \in [\tau_i, \tau_i']$. We emphasize that in the previous iteration, $g_i(t)$ is ensured to be well-refined in $I_i = [t_s, \tau_i]$, and it is now ensured in $I_i' = [t_s, \tau_i']$, where $I_i \subset I_i'$. Let $\tau_i \leftarrow \tau_i'$ and $I_i \leftarrow I_i'$ (line 14). It is what we call starting-time interval refinement. Fig. 3 illustrates the relationships between starting times and arrival times in the starting-time interval refinement.

As discussed above, the starting-time subinterval I_i for the dequeued node v_i is enlarged, while its earliest arrival-time function, $g_i(t)$, remains unchanged. Next, we discuss how to update the arrival-time function $g_j(t)$ for a node v_j, when its incoming neighbor v_i's starting-time interval is refined ($(v_i, v_j) \in E$).

Arrival-Time Function Refinement: As shown above, the arrival-time function $g_i(t)$, for node v_i, is well-refined as the earliest v_s-v_i arrival-time function in both the original starting-time subinterval, $I_i = [t_s, \tau_i]$, and the enlarged one, $I_i' = [t_s, \tau_i']$, in the previous and the current iterations, respectively. It can then be used to refine arrival-time functions, $g_j(t)$, in starting-time subinterval $[\tau_i, \tau_i']$ for all of v_i's outgoing neighbors v_j ($(v_i, v_j) \in E$). It is done in line 10-13. First, it computes the arrival time $g_j'(t)$ at v_j via edge (v_i, v_j) for starting time $t \in [\tau_i, \tau_i']$ (line 11). Then $g_j(t)$ is refined as $\min\{g_j(t), g_j'(t)\}$ on interval $[\tau_i, \tau_i']$ (line 12). We only refine $g_j(t)$ on $[\tau_i, \tau_i']$, because we have refined it with $g_i(t)$ on $[t_s, \tau_i]$ already in previous iterations. Q is updated for node v_j with its newly refined arrival-time function $g_j(t)$ (line 13).

Terminating Condition: After arrival-time function refinement, τ_i is set as τ_i' (line 14). If in the entire interval $T = [t_s, t_e]$, for node v_i, $g_i(t)$ has been well-refined, as specified in the condition "$\tau_i \geq t_e$" (line 15), the algorithm further checks whether v_i is the destination. If true, it terminates in line 17. If $g_i(t)$ has not been well-refined in the entire starting-time interval, pair $(\tau_i, g_i(t))$ is enqueued back into Q (line 19) for further time refinement.

Note: the while loop terminates when there is only one pair left in the priority queue (line 5). Let the last pair be $(\tau_i, g_i(t))$ for node v_i. There is no need to further refine it for the following reason. The starting-time subintervals I_j and earliest arrival-time functions $g_j(t)$ of all other nodes v_j have already been well refined. So $g_i(t)$ has been refined by every well-refined $g_j(t)$, if $(v_j, v_i) \in E$. Therefore, $g_i(t)$ in the starting-time interval $[\tau_i, t_e]$ also specifies the earliest arrival time, if $g_i(t) \neq \infty$ for $t \in [\tau_i, t_e]$.

In summary, for source v_s, the *timeRefinement* algorithm initializes $g_s(t) \leftarrow t$, which becomes the starting point to refine its own starting-time interval, and to refine the earliest arrival-time functions for other nodes. The starting-time interval refinement and the arrival-time function refinement repeat in every iteration.

5.3 A Running Example

Reconsider Example 2.1 to compute the query $\mathsf{LTT}(v_1, v_4, T = [0, 60])$, over G_T (Fig. 1). Algorithm 3 takes G_T, $v_s = v_1$, $v_e = v_4$, and $T = [0, 60]$ as the input. Initially, $g_1(t) = t$ (Fig. 4 (a)). It states that if it departs from v_1 at time t_0, then it arrives at v_1 at the same time t_0, and the travel time is $g_1(t_0) - t_0 = 0$ for any $t_0 \in T$. At the initial stage, the starting-time subinterval I_1 for v_1 is $[0, \tau_1]$, where $\tau_1 = 0$. The black box indicates τ_1 (x-value) and $g_1(\tau_1)$ (y-value), which states that $g_1(t)$ is ensured to specify the earliest arrival time for v_1 in $I_1 = [0, \tau_1] = [0, 0]$. For other nodes, v_i ($i = 2, 3, 4$), $g_i(t) = \infty$ and $I_i = [0, \tau_i]$ where $\tau_i = 0$. It implies that $g_i(t)$ has not been refined yet.

In the first iteration, the top pair dequeued from the priority queue Q is $(\tau_1, g_1(t))$ where $\tau_1 = 0$ and $g_t(\tau_1) = 0$ (line 6). In other words, $g_1(t)$ specifies the earliest arrival time in the starting-time subinterval $I_1 = [0, \tau_1] = [0, 0]$. It picks $(\tau_3, g_3(t))$ as $(\tau_k, g_k(t))$, where $\tau_3 = 0$ and $g(\tau_3) = \infty$ (line 7). Then, the newly enlarged starting-time subinterval, I_1', for v_1, becomes $I_1' = [0, \tau_1'] = [0, 60]$ (line 9), because τ_1' is the latest starting time t satisfying $g_1(t) \leq g_3(\tau_3) + \Delta = \infty + \infty$, where $\Delta = \infty$ as there are no coming edges to source node v_1 (line 8-9). The resulting $g_1(t)$ is shown in Fig. 4 (b). Because v_1's starting-time subinterval $I_1 = [0, 60]$, v_1 can be removed from the queue Q. In this iteration, it will update the arrival-time functions for nodes v_2 and v_3 based on v_1. The resulting arrival-time functions, $g_2(t)$ and $g_3(t)$, for v_2 and v_3, are shown in Fig. 4 (c) and (d), respectively.

In the second iteration, the top pair dequeued from queue Q is $(\tau_3, g_3(t))$, where $\tau_3 = 0$ and $g_3(\tau_3) = 5$ (line 6). It will then pick $(\tau_2, g_2(t))$ as $(\tau_k, g_k(t))$, where $\tau_2 = 0$ and $g_2(\tau_2) = 10$ (line 7). The newly enlarged starting-time subinterval, I_3', for v_3, becomes

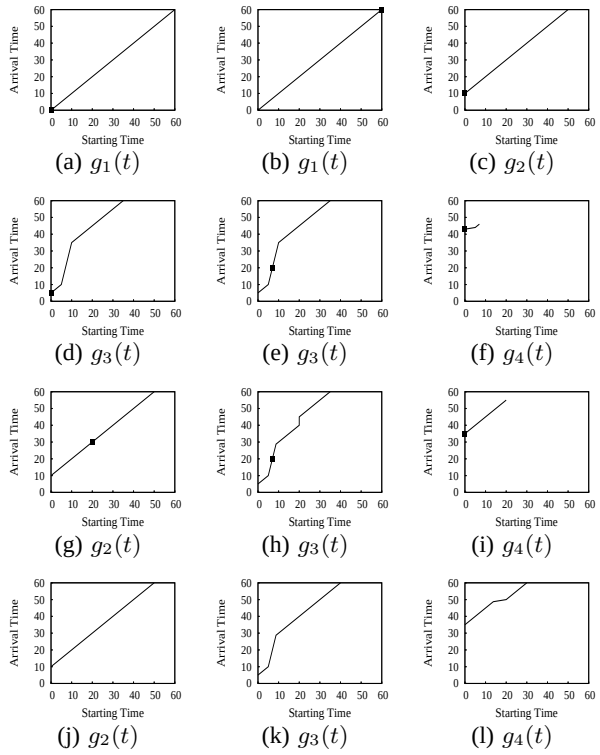


Figure 4: Arrival-Time Functions

$I'_3 = [0, \tau'_3] = [0, 7]$ (line 9), because $\tau'_3 = \max\{t | g_3(t) \leq g_2(\tau_2) + \Delta\} = \max\{t | g_3(t) \leq 20\} = 7$ (line 9), where $\Delta = \min\{w_{1,3}(g_2(\tau_2)), w_{2,3}(g_2(\tau_2))\} = \min\{w_{1,3}(10), w_{2,3}(10)\} = 10$ (line 8). The resulting $g_3(t)$ in the new starting-time subinterval $I_3 \leftarrow I'_3 = [0, 7]$ (starting-time interval refinement) is shown in Fig. 4 (e). It will also update $g_4(t)$ on $[0, 7]$ (arrival-time function refinement), which is shown in Fig. 4 (f).

In the third iteration, the top pair dequeued from the priority queue Q is $(\tau_2, g_2(t))$ where $\tau_2 = 0$ and $g_2(\tau_2) = 10$. The resulting $g_2(t)$ for v_2 is shown in Fig. 4 (g). The updated $g_3(t)$ and $g_4(t)$ are shown in Fig. 4 (h) and (i).

The iteration repeats 11 times. Functions $g_1(t)$, $g_2(t)$, $g_3(t)$, and $g_4(t)$ are well-refined, as in Fig. 4 (b), (j), (k), and (l), respectively. The optimal starting time from source v_1 is 20, and the minimized LTT is $g_4(20) - 20 = 30$. Based on functions $g_1(t) \cdots g_4(t)$, the optimal path p^* can be constructed using *pathSelection* (Algorithm 2). Let $t^* = 20$. First, in *pathSelection*, it finds that $g_3(t^*) + w_{3,4}(g_3(t^*)) = g_4(t^*)$, so the predecessor to the destination node is v_3. Second, it finds that $g_2(t^*) + w_{2,3}(g_2(t^*)) = g_3(t^*)$, so the predecessor to v_3 is v_2. Finally, in a similar fashion, it reaches the source v_1, and path p^* is identified as $(v_1, v_2)(v_2, v_3)(v_3, v_4)$.

5.4 Correctness

Theorem 5.2: *Given a FIFO time-dependent graph $G_T(V, E, W)$ and a query $\mathsf{LTT}(v_s, v_e, T)$, where $T = [t_s, t_e]$.* TWO-STEP-LTT *(Algorithm 1) finds the optimal answer to $\mathsf{LTT}(v_s, v_e, T)$.* □

Proof Sketch: As given in Theorem 5.1, there is no need to consider waiting time at nodes. TWO-STEP-LTT is a two-step algorithm, namely, *timeRefinement* algorithm and *pathSelection* algorithm. In the first step, *timeRefinement* refines the earliest arrival-time functions, $g_i(t)$, for all needed nodes v_i, to reach v_e from v_s. The optimal starting time is $t^* \in T$ which minimizes travel time

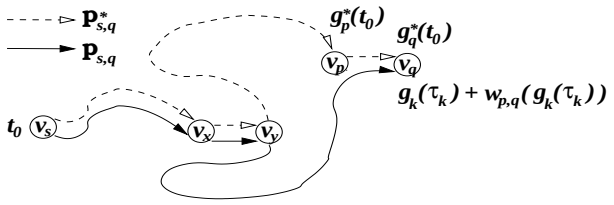


Figure 5: Intuition of the Proof

$g_e(t) - t$ from v_s to v_e. In the second step, based on t^* and arrival-time functions, $g_i(t)$, *pathSelection* recovers the optimal path p^*. The proof for the correctness of the second step is straightforward as discussed in Section 4. In Theorem 5.3, we will prove the first step is correct, to complete the proof of Theorem 5.2. □

Lemma 5.1: *For every $v_i \in V$, $g_i(t_1) \leq g_i(t_2)$, if $t_1 \leq t_2$, is always true in timeRefinement (Algorithm 3).* □

This lemma shows the monotonicity of $g_i(t)$. It can be proved directly by the definition of *FIFO* property and the way how $g_i(t)$ is initialized and updated. It will be used to prove Theorem 5.3.

Theorem 5.3: *Given a FIFO time-dependent graph $G_T(V, E, W)$ and a query $\mathsf{LTT}(v_s, v_e, T)$, where $T = [t_s, t_e]$, timeRefinement (Algorithm 3) well refines the earliest arrival-time function $g_e(t)$, which specifies the earliest arrival time at destination node v_e for starting time $t \in T$.* □

Proof: Let $g^*_i(t)$ denote the arrival-time function that specifies the earliest arrival time from v_s to v_i for starting time t. We need prove that Algorithm *timeRefinement* terminates with $g_e(t) = g^*_e(t)$ for $t \in T$ (well-refined). We prove this by proving a loop invariant: *at the beginning of every iteration of the while loop (line 6), for $v_i \in V$ and $t \in [t_s, \tau_i]$, $g^*_i(t) = g_i(t)$ is true.* Or equivalently, *after line 9 of every iteration, for $v_i \in V$ and $t \in [\tau_i, \tau'_i]$, $g^*_i(t) = g_i(t)$ is true.* The proof of this loop invariant completes the proof, because initially $\tau_i = t_s$ (line 3), and finally τ_i is greater than or equal to t_e (line 15). Therefore, with this loop invariant, when *timeRefinement* terminates, we must have $g_e(t) = g^*_e(t)$ for $t \in T = [t_s, t_e]$.

In the following, we prove the loop invariant. Based on the way how arrival time functions, $g_i(t)$, are initialized and updated, we have $g_i(t) \geq g^*_i(t)$ for every $v_i \in V$ and any $t \in T$. It suffices to show $g_i(t) \leq g^*_i(t)$ in the loop invariant, to prove $g_e(t) = g^*_e(t)$.

For the purpose of contradiction, suppose v_q is dequeued in line 6, and $g_q(t_0) > g^*_q(t_0)$, for certain starting time $t_0 \in [\tau_q, \tau'_q]$, where the loop invariant is *violated* for *the first time*. As shown in Fig. 5, let the v_s-v_q path with arrival time $g_q(t_0)$ for starting time t_0 be $\mathsf{p}_{s,q}$, and let the optimal v_s-v_q path for starting time t_0 be $\mathsf{p}^*_{s,q} = (v_s, v_t) \cdots (v_x, v_y) \cdots (v_p, v_q)$. We prove that $\mathsf{p}^*_{s,q}$ is no better than $\mathsf{p}_{s,q}$ for starting time t_0 by showing $g_q(t_0) \leq g^*_q(t_0)$.

Consider the node v_y on path $\mathsf{p}^*_{s,q}$ such that (i) $t_0 > \tau_y$ and (ii) $t_0 \leq \tau_l$ for all nodes, v_l, on the path from v_s to v_x. By (i) it means that $g_y(t_0)$ *may be not* well-refined because $t_0 \notin [t_s, \tau_y]$. We can prove actually, $g_y(t_0) = g^*_y(t_0)$ as follows: Since $t_0 \leq \tau_x$, we have $g_x(t_0) = g^*_x(t_0)$. Thus, $g_y(t_0)$ is well-refined (line 11) with $g_x(t_0)$ as $g^*_y(t_0) = g_x(t_0) + w_{x,y}(g_x(t_0))$ in previous iterations.

If $v_y = v_q$, the proof of $g_q(t_0) \leq g^*_q(t_0)$ is already completed. We will focus on the case where $v_y \neq v_q$ in the following part.

First, since v_y appears before v_p on $\mathsf{p}^*_{s,q}$ and all edge delays are nonnegative, we have

$$g_y(t_0) = g^*_y(t_0) \leq g^*_p(t_0). \tag{7}$$

Second, based on the choice of τ'_q and Δ, and the monotonicity of

$g_q(t)$ (Lemma 5.1), we have

$$\begin{aligned} g_q(t_0) &\leq g_q(\tau_q') \text{ (for } t_0 < \tau_q' \text{ and } g_q(t) \text{ is monotone)} \\ &\leq g_k(\tau_k) + \Delta \text{ (for the choice of } \tau_q' \text{ in line 9)} \\ &\leq g_k(\tau_k) + w_{p,q}(g_k(\tau_k)) \qquad (8) \\ &\quad \text{(for the choice of } \Delta \text{ in line 8, note } (v_p, v_q) \in E) \end{aligned}$$

Third, because of the choice of $(\tau_k, g_k(t))$ in line 7, $g_k(\tau_k)$ is the second earliest arrival time in Q following $g_q(\tau_q)$. We have

$$g_k(\tau_k) \leq g_y(\tau_y). \qquad (9)$$

Fourth, because of the choice of v_y ($t_0 > \tau_y$) and the monotonicity of $g_y(t)$ (Lemma 5.1), we have

$$g_y(\tau_y) \leq g_y(t_0). \qquad (10)$$

Then, based on Equation (9), Equation (10), and Equation (7), we have

$$g_k(\tau_k) \leq g_y(\tau_y) \leq g_y(t_0) \leq g_p^*(t_0). \qquad (11)$$

Recall G_T is a *FIFO* graph. Based on Equation (11) and the *FIFO* property of edge (v_p, v_q), we have

$$g_k(\tau_k) + w_{p,q}(g_k(\tau_k)) \leq g_p^*(t_0) + w_{p,q}(g_p^*(t_0)). \qquad (12)$$

Note: $\mathsf{p}_{s,q}^*$ is the optimal path for the starting time t_0. We have $g_q^*(t_0) = g_p^*(t_0) + w_{p,q}(g_p^*(t_0))$. Based on Equation (8) and Equation (12), we can conclude that $g_q(t_0) \leq g_p^*(t_0) + w_{p,q}(g_p^*(t_0)) = g_q^*(t_0)$, which completes the proof. □

5.5 Time/Space Complexity

In this subsection, we give time complexity for manipulating piecewise-linear functions followed by the time/space complexity of our algorithms.

Representing Functions: Let $f(t)$ and $g(t)$ be piecewise-linear functions, defined on a time interval $T = [t_s, t_e]$, and suppose that $f(t)$ and $g(t)$ can be represented as p and q pieces of linear functions on subintervals of T, respectively, such that $f(t) = \langle (f_1, t_1^f), (f_2, t_2^f), \cdots, (f_p, t_p^f) \rangle$ and $g(t) = \langle (g_1, t_1^g), (g_2, t_2^g), \cdots, (g_q, t_q^g) \rangle$. Each pair (f_i, t_i^f) represents a linear function $f_i(t)$ on the subinterval $[t_i^f, t_{i+1}^f)$, and each pair (g_i, t_i^g) represents a linear function $g_i(t)$ on the subinterval $[t_i^g, t_{i+1}^g)$. Note: we let $t_1^f = t_1^g = t_s$ and $t_{p+1}^f = t_{q+1}^g = t_e$, where t_s and t_e are the two ends of T. General functions can be represented in a similar way.

Implementing Function Operations: Given two such functions $f(t)$ and $g(t)$, let a and b be two constants. Four operations are defined and used in our algorithms, namely, FUNCTION INVERSE, $f^{-1}(a) \triangleq \max\{t | f(t) = a\}$, LINEAR COMBINATION, $a \cdot f(t) + b \cdot g(t)$, FUNCTION COMPOUND, $f(g(t))$, and MINIMUM of two functions, $\min\{f(t), g(t)\}$. Each operation outputs a piecewise-linear function. The time complexity for the function inverse is $O(p)$ by swapping p pairs of (f_i, t_i^f). The time complexity for the other three operations is $O(p + q)$ by sweeping each of the two sequences of pairs only once. In addition, the function value $f(t_0)$ for a given time instance t_0 ($t_s \leq t_0 \leq t_e$) can be computed in $O(\log p)$ time using binary search. Details are omitted here.

In the following analysis of algorithm TWO-STEP-LTT, we take the cost of function operations into consideration, and use the similar notations for functional complexity used in [20]. We use $\alpha(T)$ or $\alpha(|T|)$ to denote the time/space complexity of maintaining a piecewise-linear function, or manipulating a function operation, defined in time interval T ($|T|$ is the length of T). Based on the representation and implementation of functions as introduced above, the time/space required for a function operation is linearly proportional to the number of pieces needed to represent the function in T. So if assuming the number of pieces needed is linearly proportional to $|T|$, we have $O(\alpha(|T_1 \cup T_2|)) = O(\alpha(|T_1|)) + O(\alpha(|T_2|))$ for $T_1 \cap T_2 = \varnothing$. This assumption is used in our following analysis of complexity. Note algorithms, BELLMAN-FORD, KDXZ, and our TWO-STEP-LTT, manipulate functions in the same manner.

Complexity of Two-Step-LTT: Given a graph G_T with n nodes and m edges in total, consider query $\mathsf{LTT}(v_s, v_e, T)$.

Lemma 5.2: *The time complexity of timeRefinement (Algorithm 3) is* $O((n \log n + m)\alpha(T))$. □

Proof Sketch: In each iteration of the while loop, the priority queue Q of length at most n is accessed in line 6, 7, 13, and 19. Using Fibonacci Heap [4], both $dequeue(Q)$ and $head(Q)$ require $O(\log n)$ amortized time, and both $update(Q, (\tau_j, g_j(t)))$ (when $g_j(t)$ is updated) and $enqueue(Q, (\tau_i, g_i(t)))$ (when the new pair $(\tau_i, g_i(t))$ is inserted) require $O(1)$ amortized time. Moreover, line 8 requires $O(d_i)$ time to find Δ, where d_i is the in-degree of node v_i, and line 9 requires $O(\alpha(\tau_i' - \tau_i))$ time to find τ_i'. In line 11-12, the arrival-time functions, $g_j(t)$, are refined within the starting-time subinterval $[\tau_i, \tau_i']$ in $O(\alpha(\tau_i' - \tau_i))$ time. Therefore, for each iteration of the while loop, it needs $O(\log n + d_i + d_i\alpha(\tau_i' - \tau_i)) \leq O((\log n + d_i)\alpha(\tau_i' - \tau_i))$ time.

Let $\tau_i^{(k)}$ denote the value of τ_i when v_i is dequeued from Q for the k^{th} time, and let l_i denote the number of times that v_i is dequeued from Q in total ($k \leq l_i$). Then the total time complexity is $O(\sum_{v_i \in V} \sum_{k=1}^{l_i} ((\log n + d_i)\alpha(\tau_i^{(k)} - \tau_i^{(k-1)})))$. Because

$$\begin{aligned} & \sum_{k=1}^{l_i} ((\log n + d_i)\alpha(\tau_i^{(k)} - \tau_i^{(k-1)})) \\ = \; & (\log n + d_i)\alpha(\sum_{k=1}^{l_i} (\tau_i^{(k)} - \tau_i^{(k-1)})) \\ = \; & (\log n + d_i)\alpha(|t_e - t_s|) = (\log n + d_i)\alpha(T), \end{aligned}$$

the total time complexity is:

$$O(\sum_{v_i \in V} (\log n + d_i)\alpha(T)) = O((n \log n + m)\alpha(T)). \qquad \square$$

Lemma 5.3: *The time complexity of pathSelection (Algorithm 2) is* $O(m\alpha(T))$. □

Proof Sketch: Because the value of $g_j(t^*)$ is strictly decreasing in every iteration, every node $v_j \in V$ will be examined at most once in *pathSelection* (line 5). Let d_j denote the in-degree of node v_j. The while loop requires $O(d_j\alpha(T))$ time for each v_j. The time complexity of *pathSelection* can be computed as $O(\sum_{v_j \in V} d_j\alpha(T)) = O(m\alpha(T))$. □

From the above two lemmas (Lemma 5.2 and 5.3), we can prove the time complexity of TWO-STEP-LTT.

Theorem 5.4: *The time complexity of* TWO-STEP-LTT *(Algorithm 1) is* $O((n \log n + m)\alpha(T))$. □

In both *timeRefinement* and *pathSelection*, we need maintain graph $G(V, E, W)$ with m edges and m functions, $w_{i,j}(t)$, for $(v_i, v_j) \in E$. During the execution of algorithms, we need maintain a priority queue Q with at most n elements, and n arrival-time functions, $g_i(t)$, for $v_i \in V$. Therefore, the total space complexity is $O((n + m)\alpha(T))$.

Theorem 5.5: *The space complexity of* TWO-STEP-LTT *(Algorithm 1) is* $O((n + m)\alpha(T))$. □

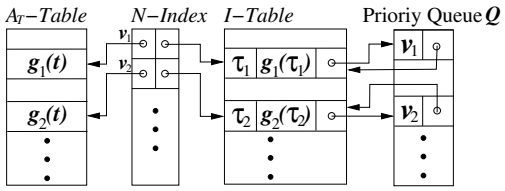


Figure 6: Runtime Data Structures

5.6 Storage Model and Implementation

In this subsection, we give some implementation details of our solution. A time-dependent graph $G_T(V, E, W)$ is maintained using an edge representation, where an edge is stored as a triple $(v_i, v_j, w_{i,j}(t))$. The edges can be stored in a table. The first column (v_i) and the second column (v_j) are fixed-size whereas the third column (edge-delay function) is variable-size. Two B+-trees are built on the top of the table. One is built on the first column (v_i), and the other is built on the second column (v_j). They can efficiently support all the necessary operations w.r.t. G_T.

For a given $\mathsf{LTT}(v_s, v_e, T)$ where $T = [t_s, t_e]$, as shown in Fig. 6, TWO-STEP-LTT needs to maintain four runtime data structures, namely, *N-Index* (a list of node identifiers), *A_T-Table* (a list of arrival-time functions $g_i(t)$'s, for all $v_i \in V$), *I-Table* (a list of pairs $(\tau_i, g_i(\tau_i))$'s, for all $v_i \in V$), and the priority query Q.

The *N-Index* is a list of two pointers, which is sorted by node identifiers in order to be accessed efficiently. Given a node v_i, one of the two pointers in *N-Index* points to its arrival-time function, $g_i(t)$, which is separately maintained in *A_T-Table*. The arrival-time functions are maintained separately from *N-Index*, because they are variable-size. The separation also allows *N-Index* to remain unchanged, when arrival-time functions need to be updated. The other pointer in *N-Index* points to *I-Table* where $(\tau_i, g_i(\tau_i))$'s are maintained. Given v_i, it allows us to quickly find the corresponding pair $(\tau_i, g_i(\tau_i))$, when it needs to be updated.

The priority queue Q sorts $(\tau_i, g_i(t))$ in the ascending order of $g_i(\tau_i)$. Every element of Q in our storage model is a pointer pointing to the corresponding pair $(\tau_i, g_i(\tau_i))$ which is maintained in *I-Table*. From each element in *I-Table*, there is also a pointer pointing back to the position of the pair in Q. Such implementation is designed to reduce the size of Q in the running-time storage.

Since *N-Index* and priority queue Q are not large, they can be maintained in memory. When G_T is too large to be stored in the main memory, we maintain *A_T-Table*, *I-Table*, and G_T on disk.

5.7 Solution for Non-FIFO Graphs

In this subsection, we discuss how to find the optimal LTT over a (general) non-*FIFO* time-dependent graph. We show that we can transform such a non-*FIFO* graph $G'_T(V, E, W')$ into a *FIFO* graph $G_T(V, E, W)$ where both V and E remain unchanged. Then we can process $\mathsf{LTT}(v_s, v_e, T)$ on the *FIFO* graph G_T using our proposed TWO-STEP-LTT algorithm. The optimal path p^* found in G_T can be converted into an optimal path p'^* in the original non-*FIFO* graph G'_T, by inserting some waiting time on each node in path p^*. The similar idea was also used in [20].

For each edge-delay function $w'_{i,j}(t)$ in the non-*FIFO* graph G'_T, we define $w_{i,j}(t)$ to construct a *FIFO* graph G_T.

$$\begin{aligned} w_{i,j}(t) &= \Delta_{i,j}(t) + w'_{i,j}(t + \Delta_{i,j}(t)) \\ &= \min_{0 \le t_\Delta \le t_e - t} \{t_\Delta + w'_{i,j}(t + t_\Delta)\} \end{aligned} \quad (13)$$

Since the starting-time interval $T = [t_s, t_e]$ is a closed interval, $w_{i,j}(t)$ and $\Delta_{i,j}(t)$ in Equation (13) are well-defined. Intuitively, $\Delta_{i,j}(t)$ is the optimal waiting time to traverse edge (v_i, v_j), if arriving at v_i at time t. If there are multiple possible values of t_Δ to minimize $w'_{i,j}(t + t_\Delta) + t_\Delta$, we select any of them as $\Delta_{i,j}(t)$. It is easy to verify that edge (v_i, v_j) with edge delay function $w_{i,j}(t)$ has the *FIFO* property. Let W be the set of newly defined edge delays $w_{i,j}(t)$'s, and then $G_T(V, E, W)$ is a *FIFO* graph.

Suppose using TWO-STEP-LTT algorithm, we find the optimal path $\mathsf{p}^* = (v_1, v_2) \cdots (v_{k-1}, v_k)$, where $v_1 = v_s$ and $v_k = v_e$, together with the best starting time $t^* \in T$, for $\mathsf{LTT}(v_s, v_e, T)$ on the converted *FIFO* graph $G_T(V, E, W)$. We construct the optimal path p'^* for $\mathsf{LTT}(v_s, v_e, T)$ on the original non-*FIFO* graph $G'_T(V, E, W')$ by inserting waiting time $\varpi^*(v_i) = \Delta_{i,i+1}(t)$ at node v_i for $1 \le i \le k-1$, where t is the arrival time at node v_i along path p^* in G_T for starting time t^*.

6. PERFORMANCE STUDIES

In this section, we conducted extensive experimental studies to compare our solution, TWO-STEP-LTT, with other three algorithms for the TDSP problem, namely, the most efficient discrete-time algorithm DOT [2], BELLMAN-FORD based algorithm OR [20], and A* algorithm KDXZ [15]. We implemented all algorithms using C++. Note we denote our TWO-STEP-LTT as "2S" in figures below for conciseness. For DOT, let δ be the length of the interval between two adjacent time points, and we used $\delta = 0.1$ (unit).

Experiment Setup: We use a real dataset with 16,326 nodes and 26,528 edges, representing the road-map in the Maryland State in US. The dataset is extracted from the US Census Bureau 2005 TIGER/Line[4]. The nodes represent the starts, ends, and intersections of roads, while the edges represent road segments. Note the four algorithms can handle both undirected and directed graphs. In experiments, we represent the real database as a directed time-dependent graph G_T. We further generate 10 subgraphs $G_1, \cdots, G_{10}$ from G_T with the number of nodes varying from 40 to 10K. Each subgraph corresponds to a subarea of G_T. The numbers of nodes and edges of $G_1 \cdots G_{10}$ are listed in Table 2.

We test the class of continuous piecewise-linear edge-delay functions $W = \{w_{i,j}(t)\}$, whose operations are implemented as described in Section 5.5. Each $w_{i,j}(t)$ is generated randomly with four parameters, *average-delay* $\bar{w}$, *range-delay* w_Δ, *length-domain* $L_\mathcal{T}$, and *number-segment* $N_\mathcal{T}$, in domain $\mathcal{T} = [0, L_\mathcal{T}]$ independently as follows: $\mathcal{T}$ is randomly divided into $N_\mathcal{T}$ subintervals; within each one, $w_{i,j}(t)$ is a linear function; the value of $w_{i,j}(t)$ at the start/end of each subinterval is randomly generated as a number in $[\bar{w} - w_\Delta, \bar{w} + w_\Delta]$ uniformly. Note: the $w_{i,j}(t)$'s generated as described above are general edge-delay functions (some may have *FIFO* properties while the others may be non-*FIFO*).

In Experiment-1 and Experiment-2, $\bar{w} = 11$, $w_\Delta = 9$, $L_\mathcal{T} = 2,000$, and $N_\mathcal{T}$ is randomly picked from 4 to 8. In Experiment-3, we will vary some of the four parameters to test the scalability of algorithms w.r.t. different types of edge-delay functions.

The set of queries, $\{\mathsf{LTT}(v_s, v_e, T)\}$, used in each test is constructed by fixing source v_s as the center of graph, and varying destination v_e over all the other nodes in graph.

We conducted all tests on a 2.8GHz CPU/1G memory PC running XP. We report the processing time (second), and the memory consumed (byte).

Experiment-1 (Graph-Scalability): For query $\mathsf{LTT}(v_s, v_e, T)$, we fix starting-time interval $T = [0, 500]$, and vary (i) the number of nodes, and (ii) the number of edges, in the time-dependent graph.

[4]Topologically Integrated Geographic Encoding and Referencing system: http://www.census.gov/geo/www/tiger/

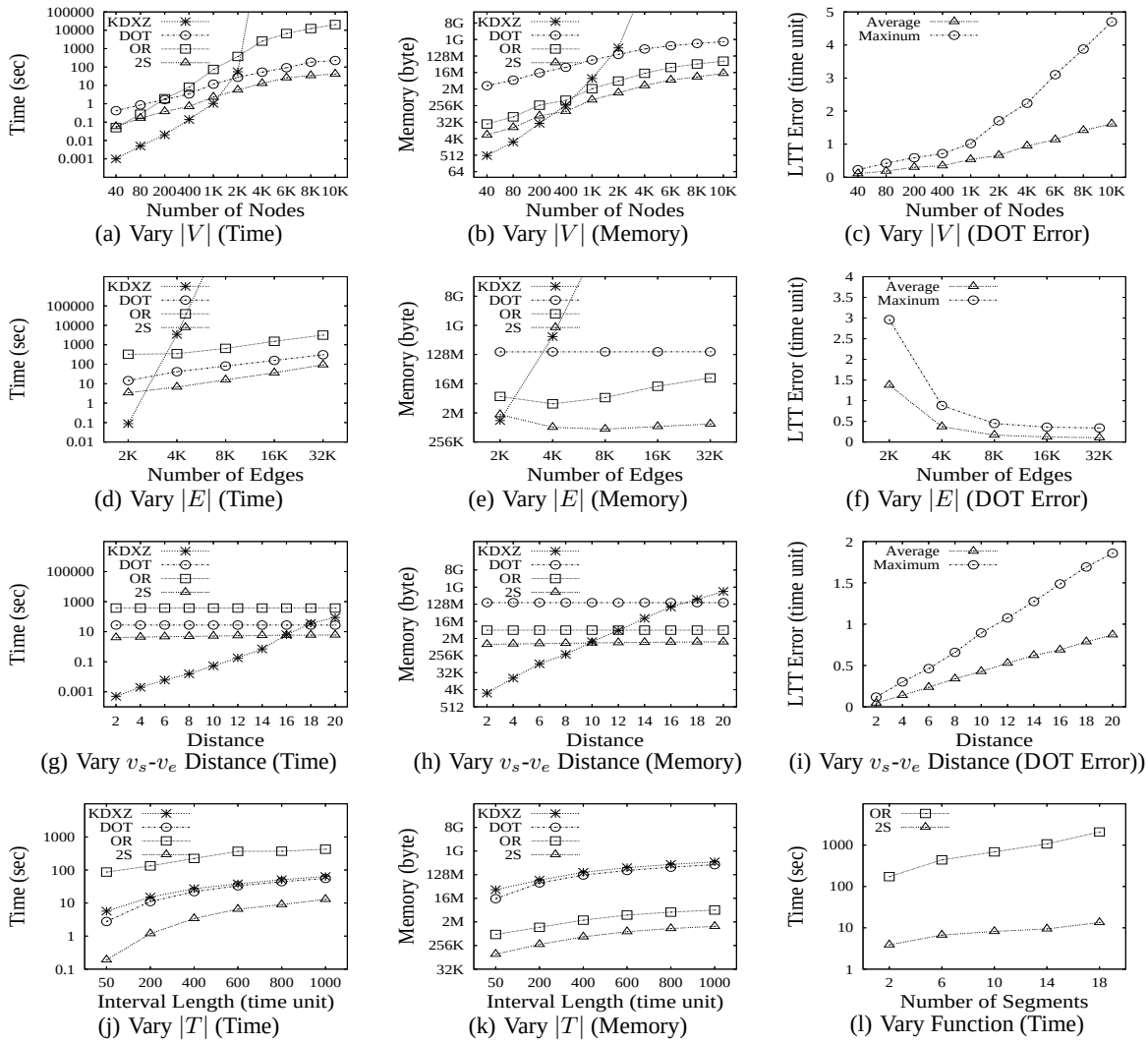


Figure 7: Testing Algorithms for TDSP Problem

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}
$\|V\|$	40	80	200	400	1K	2K	4K	6K	8K	10K
$\|E\|$	52	107	262	548	1.5K	3K	6.4K	9.7K	13K	16K

Table 2: Datasets

First, with the number of nodes increasing from 40 to 10K in $G_1, \cdots, G_{10}$, the average processing time and memory consumed are shown in Fig. 7(a) and Fig. 7(b). The average processing time of DOT is about 10 times larger than our TWO-STEP-LTT consistently. TWO-STEP-LTT outperforms OR when the number of nodes is > 80. KDXZ algorithm is the fastest when there are less than 1K nodes, but it becomes slower than TWO-STEP-LTT when there are more than 1K nodes. The average processing time of KDXZ increases exponentially, and it cannot finish for most queries in this experiment when the number of nodes is $>$ 4K, because its search space is exponentially w.r.t. the size of graph. TWO-STEP-LTT outperforms OR significantly when the graph is large in size. The average memory consumed by OR is 5 times larger than TWO-STEP-LTT, because OR maintains two sets of functions, $\{g_i(t)|v_i \in V\}$ and $\{h_{j,k}(t)|(v_j, v_k) \in E\}$, while TWO-STEP-LTT only maintains $\{g_i(t)|v_i \in V\}$. The memory consumed by DOT is 50 times larger than TWO-STEP-LTT consistently. KDXZ consumes much more than TWO-STEP-LTT does when the number of nodes is $>$ 1K. Note: KDXZ, OR, and TWO-STEP-LTT find the optimal LTT. Fig. 7(c) shows the average and maximum LTT error of DOT. By LTT error, we mean the difference between the LTT obtained by DOT and the optimal LTT. When the number of nodes increases, LTT error becomes larger, because the average distance between two nodes becomes larger.

Second, we vary the density of G_6 by fixing the number of nodes as 2K while changing the number of edges. 5 graphs are generated with 2K, 4K, 8K, 16K and 32K edges. We report average processing time and memory consumed in Fig. 7(d) and 7(e) respectively. The average processing time increases when the number of edges increases. TWO-STEP-LTT significantly outperforms DOT and KDXZ in all the cases, and outperforms KDXZ when the number of edges is $>$ 2K. KDXZ cannot find LTT in reasonable time for most queries tested when the number of edges is $>$ 4K. In terms of memory consumption, TWO-STEP-LTT performs the best followed by OR and then DOT. The amount of memory KDXZ consumes is exponentially proportional to the number of edges. Fig. 7(f) shows the average/maximum LTT error of the DOT. The error becomes smaller when the number of edges increases, because with more edges, the average distance between

two nodes becomes smaller, which decreases the LTT error.

Experiment-2 (Query-Scalability): We use G_6 (with 2K nodes and 3K edges). For query $\mathsf{LTT}(v_s, v_e, T)$, (i) we vary the number of nodes on the shortest v_s-v_e path (i.e., v_s-v_e *distance*, x-axis "Distance" in Fig. 7(g)-7(i)); (ii) we change the length of starting-time interval T (i.e., x-axis "Interval Length" in Fig. 7(j)-7(k)).

First, the starting-time interval T is fixed to be $[0, 500]$ for all queries. With the number of nodes on the shortest v_s-v_e path increasing from 2 to 20, we report the average processing time and memory consumed in Fig. 7(g) and 7(h) respectively. Note the time/memory consumed by DOT and OR are nearly unchanged, because they cannot terminate until the LTT from v_s to every other node is determined. KDXZ performs well if the v_s-v_e distance is < 15, but quickly deteriorates otherwise. TWO-STEP-LTT constantly outperforms KDXZ after the v_s-v_e distance is > 16 on both time and memory consumed, because the size of the search space in KDXZ increases exponentially w.r.t. v_s-v_e distance. Fig. 7(i) shows the average/max LTT error of DOT which becomes larger, while the number of nodes on the v_s-v_e shortest path increases.

Second, we vary the length of starting-time interval T from 50 to 1,000, and report the average of processing time and memory consumed in Fig. 7(j) and 7(k) respectively. It is shown that all algorithms need additional time/memory with the length of interval T increases, because the increment of $|T|$ incurs both additional function-operation time and search space. TWO-STEP-LTT outperforms the others consistently.

Experiment-3 (Edge-Delay Function): We test the effect of edge-delay functions on the processing time of TWO-STEP-LTT and OR, since both request larger numbers of function operations than the other two. We use G_6, and fix $T = [0, 500]$.

First, for every edge, we fix $\bar{w} = 11$, $w_\Delta = 9$, $L_\mathcal{T} = 2000$, and vary $N_\mathcal{T}$ from 2 to 18. When $N_\mathcal{T}$ increases, the edge delay function fluctuates more frequently. We report the average processing time consumed by TWO-STEP-LTT and OR in Fig. 7(l). TWO-STEP-LTT outperforms OR.

Second, we fix $\bar{w} = 11$, $L_\mathcal{T} = 2000$, $N_\mathcal{T} = 8$, and vary w_Δ from 2 to 10. When w_Δ increases, both TWO-STEP-LTT and OR consume more time, because resulting functions $g_i(t)$'s can be more complicated, and hence require more function-operation time in both. TWO-STEP-LTT again outperforms OR in this test. Due to the limit of space, we do not report the detailed result of this test.

7. RELATED WORK

As shown in [26], answering $\mathsf{LTT}(v_s, v_e, t)$ for a given starting time t (not a starting-time interval) in a time-dependent graph G_T can be solved similarly as a single-source shortest-path problem in a static graph with constant edge delays [26]. Chon et al. in [3] proposed a system architecture to answer $\mathsf{LTT}(v_s, v_e, t)$ for a given starting time t in a distributed environment. The variations of single-source shortest-path problem and related issues have been intensively studied in the areas of transportation [15, 12, 13, 17, 2, 29, 21], navigation systems [24, 14, 3, 9], and networks [18, 19].

Dynamic Shortest-Path (DSP): The *DSP* problem is to recompute shortest-paths repeatedly, while the underneath graph with constant edge delays is allowed to be updated from time to time. The updates include insertion/deletion of edges and edge-weight updates. Frigioni et al. in [8] proposed fully dynamic algorithms with optimal space requirements and query time for single-source *DSP* problem, and King in [16] presented the first fully dynamic algorithm for all-pair *DSP* problem. Demetrescu and Italiano in [6] presented an improved all-pair *DSP* algorithm, which can find the shortest paths in optimal worst-case time. Experimental evaluations for *DSP* algorithms can be found in [7, 5]. Roditty discussed the hardness of the *DSP* problems in [22]. The TDSP problem studied in this paper deals with edge-delay functions over a fixed time-dependent graph, whereas *DSP* deals with unpredictable updates against a static graph. They are two different problems and the techniques used in one cannot be directly applied to the other.

Hierarchy-Based Method: In order to deal with a large graph, hierarchy-based methods partition the graph into small fragments and materialize the shortest-paths between border nodes in different fragments. The shortest-path between two nodes in the graph is obtained by combining the shortest-paths from different fragments [12, 13, 14]. Different graph partitioning methods for the shortest-path problem were studied, such as disjoint edge-set partition [12, 13] and disjoint node-set partition [14]. Shekhar et al. in [23] studied the materialization trade-offs. [27] proposed a linear-time algorithm for the static single-source shortest-path problem using graph partitioning idea. Graph-partition techniques can be also embedded into our algorithm to find LTT over time-dependent graphs.

Storage of Graph and I/O Efficiency: Shekhar et al in [25] proposed *CCAM* (Connectivity-Clustered Access Method), and studied how to store large graphs on disk using connectivity clustering and to support basic operations, such as insert, delete, create, find, and get-successor, which are necessary for most graph algorithms (including our algorithm presented in this paper). Graph update is also discussed in [25]. Huang et al. in [10] studied spatial partition clustering which creates balanced partitions of links based on the spatial proximity of nodes. Woo et al. in [28] studied network traversal clustering for the storage of graphs based on graph partitioning. For the shortest-path problem, Jiang in [11] analyzed the I/O-efficiency of several representative algorithms, and their properties regarding database applications. Experimental results regarding I/O-efficiency of shortest-path algorithms can be found in [24].

8. CONCLUSIONS

In this paper, we studied the time-dependent shortest-path problem, that is, answering query $\mathsf{LTT}(v_s, v_e, T)$ in a time-dependent graph $G_T(V, E, W)$. We proposed a new DIJKSTRA-based algorithm to find the optimal $\mathsf{LTT}(v_s, v_e, T)$ with time complexity $O((n \log n + m)\alpha(T))$ and space complexity $O((n + m)\alpha(T))$, where n is the number of nodes, m is the number of edges, and $\alpha(T)$ is the cost required for each function operation. We conducted extensive studies over large time-dependent graphs, and confirmed that our algorithm can obtain the optimal $\mathsf{LTT}(v_s, v_e, T)$ efficiently for handling large time-dependent graphs.

Acknowledgment: The authors thank Donghui Zhang and Yang Du for the helpful discussions and the code provided. This work was supported by a grant of RGC, Hong Kong SAR, China (No. 418206).

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APPENDIX

A. ABOUT ROAD NETWORK

Kanoulas et al. in [15] studied finding the optimal LTT over a road network with speed-patterns, which is defined as a graph $G_S(V, E, L, S)$: $V = \{v_i\}$ is a set of nodes; $E \subseteq V \times V$ is a set of edges; L is a set of edge lengths; S is a set of speed-pattern functions. Each edge (road) $(v_i, v_j) \in E$ is associated with a length $l_{i,j} \in L$, and a speed-pattern function $\mathbf{s}_{i,j}(t) \in S$. The speed of all vehicles on edge (v_i, v_j) is at most $\mathbf{s}_{i,j}(t)$ at time t in domain $\mathcal{T}$.

For finding LTT over a road network, Kanoulas et al. construct an equivalent time-dependent graph $G_T(V, E, W)$ from the road network $G_S(V, E, L, S)$, where the node/edge set (V and E) of G_S is the same as the node/edge set of G_T. The relationship between $w_{i,j}(t)$ and $(l_{i,j}, \mathbf{s}_{i,j}(t))$ is given below, where t is the departure time from v_i.

$$w_{i,j}(t) = \min\{w | \int_t^{t+w} \mathbf{s}_{i,j}(z)dz = l_{i,j}\} \quad (14)$$

To show time-dependent graph G_T constructed from road network G_S is an *FIFO* graph, we only need to show, for $w_{i,j}(t)$ defined in Equation (14), $w_{i,j}(t_0) \leq t_\Delta + w_{i,j}(t_0 + t_\Delta)$ for $t_0 \in \mathcal{T}$ and $t_\Delta \geq 0$ (Definition 5.1). For the purpose of contradiction, we suppose $w_{i,j}(t_0) = t_\Delta + w_{i,j}(t_0 + t_\Delta) + \epsilon$ for $\epsilon > 0$. Based on Equation (14), for departure time $t = t_0$ and $t = t_0 + t_\Delta$, we have

$$l_{i,j} = \int_{t_0}^{t_0 + w_{i,j}(t_0) = t_0 + (t_\Delta + w_{i,j}(t_0 + t_\Delta) + \epsilon)} s_{i,j}(z)dz \quad (15)$$

and

$$l_{i,j} = \int_{(t_0 + t_\Delta)}^{(t_0 + t_\Delta) + w_{i,j}(t_0 + t_\Delta)} s_{i,j}(z)dz, \quad (16)$$

respectively. From Equation (15)-(16) and $s_{i,j}(z) \geq 0$, we must have $s_{i,j}(z) = 0$ for $z \in [t_0 + t_\Delta + w_{i,j}(t_0 + t_\Delta), t_0 + t_\Delta + w_{i,j}(t_0 + t_\Delta) + \epsilon]$, and thus from Equation (15), we have

$$l_{i,j} = \int_{t_0}^{t_0 + (t_\Delta + w_{i,j}(t_0 + t_\Delta))} s_{i,j}(z)dz. \quad (17)$$

From the definition of $w_{i,j}(\cdot)$ in Equation (14), the above equation contradicts with $w_{i,j}(t_0) = t_\Delta + w_{i,j}(t_0 + t_\Delta) + \epsilon$. Therefore, $w_{i,j}(t_0) \leq t_\Delta + w_{i,j}(t_0 + t_\Delta)$, and time-dependent graph G_T constructed from road network G_S is an *FIFO* graph.